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# Boundary critical phenomena in $SU(3)$ ‘spin’ chains

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## Abstract

$SU(3)$ -invariant ‘spin’ chains with a single impurity, such as a modified exchange coupling on one link, are analysed using boundary conformal field theory techniques. These chains are equivalent to a special case of the ‘ $tJV$ ’ model, i.e. the  $tJ$  model with a nearest-neighbour repulsion added. In the continuum limit they are equivalent to two free bosons at a special value of the compactification radii. The  $SU(3)$  symmetry, which is made explicit in this formulation, provides insight into the exact solution of a non-trivial boundary critical point found earlier in another formulation of this model as a theory of quantum Brownian motion.

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## 1. Introduction

Recently, there has been considerable interest in conformal field theory (CFT) with boundaries in the context of open string theory, classical statistical mechanics and quantum impurity problems in condensed matter physics. Of particular interest are certain non-trivial boundary critical points first discovered [1, 2] in the context of a constriction in a quantum wire. While the continuum limit of these models is simply two free bosons (one for charge and one for spin), the non-trivial boundary critical points *do not* correspond to any variant of simple Dirichlet (D) and Neumann (N) boundary conditions. They correspond to boundary critical points with intermediate (neither 0 nor 1) transmission amplitude through the constriction. The original approach of Kane and Fisher [2] was only able to study them using a type of  $\epsilon$ -expansion around limiting values of the compactification radii or bulk interaction parameters where they became trivial. Later Yi and Kane [3] reinvestigated these critical points in the context of a model of quantum Brownian motion on a triangular lattice finding a special value of compactification radii where the non-trivial critical point could be solved exactly. This special point was very recently investigated by the present authors [4] using the CFT techniques of conformal embedding and fusion, relating it to the three-state Potts model with a boundary [5]. However, a general solution for these non-trivial boundary critical points, for all values of the

compactification radii where they occur is still lacking. More generally, a framework seems to be lacking for understanding non-trivial boundary critical points in multi-component free boson theories.

The purpose of the present work is to provide yet another view of the special soluble point. In this case the microscopic formulation is an  $SU(3)$  ‘spin’ chain with the objects transforming under the fundamental representation of  $SU(3)$  at each lattice site and a permutation Hamiltonian. We will show that the  $SU(3)$  symmetry is sufficient to uniquely pick out the solvable non-trivial critical point without any fine-tuning. It is convenient to introduce three fermion annihilation operators  $\eta_{j\alpha}$ , on each site  $j$ , with  $\alpha = 0, 1, 2$  and a single-occupancy constraint:

$$\eta_j^{\dagger\alpha} \eta_{j\alpha} = 1. \quad (1.1)$$

We use a superscript for annihilation operators and repeated indices (one upper and one lower) are always summed. The permutation Hamiltonian can then be written as

$$H = \frac{1}{2} \sum_j J_j \eta_j^{\dagger\alpha} \eta_{\beta j} \eta_{j+1}^{\dagger\beta} \eta_{j+1\alpha}. \quad (1.2)$$

This is the three-component Lai–Sutherland model [6] in the case where all three objects obey fermionic statistics. (The same model is obtained if all three objects obey bosonic statistics.) This model is equivalent to a special case of the ‘ $tJV$ ’ model, i.e. the  $tJ$  model with an additional nearest-neighbour repulsion, as we review in section 4. This model is Bethe ansatz integrable and has a gapless excitation spectrum. Its continuum limit [10] is the  $SU(3)_1$  Wess–Zumino–Witten (WZW) nonlinear  $\sigma$ -model. We will study this model in the case where one or more links have a modified exchange coupling  $J_j$ . All other links have a fixed antiferromagnetic exchange coupling  $J > 0$ . The behaviour is quite different than that of the corresponding  $SU(2)$  spin chains (with  $S = 1/2$ ). In the  $SU(2)$  case modifying one link produces a renormalization group (RG) flow to an open chain fixed point, corresponding to the exchange coupling on the modified link renormalizing to 0 (or  $\infty$ ). However, in the  $SU(3)$  case a flow instead occurs to a non-trivial fixed point which does not correspond to an open chain nor to a uniform chain. This corresponds to the intermediate transmission coefficient fixed point [1,2] in the  $tJV$  formulation. The  $SU(3)$  symmetry of the model makes it possible to study the fixed point using fusion. Our approach is to first regard the right-movers as a second branch of left-movers, reflecting them at the impurity location. The two copies of left-moving  $SU(3)_1$  WZW excitations can then be represented by the conformal embedding

$$SU(3)_1 \times SU(3)_1 \equiv SU(3)_2 \times \text{Potts}. \quad (1.3)$$

This corresponds to the sum of central charges

$$2 + 2 = 16/5 + 4/5. \quad (1.4)$$

The non-trivial critical point can be reached by fusion either in the  $SU(3)_2$  or Potts sector. We note that the original solution of this model by Yi and Kane [3] mapped it onto the three-channel  $SU(2)$  Kondo problem, corresponding to the  $SU(2)_3$  WZW model. This model is related by a duality transformation to the  $SU(3)_2$  WZW model. We show that the spinful Luttinger liquid model at the value of the bulk interaction parameters,  $g_\sigma = 2$ ,  $g_\rho = 2/3$ , where the non-trivial critical point can be studied exactly, has an  $SU(3)$  symmetry. This provides the most natural understanding of what is special about this point in parameter space.

In the next section we review the analogous boundary critical phenomena in the ordinary ( $S = 1/2$ )  $SU(2)$  chain. Section 3 contains our new results on the  $SU(3)$  chain. Section 4 discusses the connection with the  $tJV$  model. Section 5 discusses more general models where the  $SU(3)$  symmetry is broken down to  $SU(2) \times U(1)$ , either at the boundary only, or also

in the bulk. In Section 6 we discuss the connection of the  $SU(3)$  spin chain boundary critical behaviour with that which occurs in the two-channel  $SU(3)$  Kondo model and in the triangular lattice quantum Brownian motion model. We also comment on the extension of this work to the general  $SU(N)$  case.

## 2. $SU(2)$ case

Here we consider an ordinary  $SU(2)$   $S = 1/2$  chain with a single impurity. The Hamiltonian is written as

$$H = \sum_j J_j \mathbf{S}_j \cdot \mathbf{S}_{j+1}. \quad (2.1)$$

The exchange couplings are  $J_j = J$  on all links except for one where  $J_0 = J'$  or two neighbouring links where  $J_0 = J_1 = J'$ . It was argued in [7] that the only fixed points that occur in this problem correspond to the uniform and open chain. Modifying one link leads to a flow to the open chain fixed point. If  $J' < J$ , we may think of  $J'$  as renormalizing to 0, corresponding to an open chain. If  $J' > J$ , we may think of  $J'$  as renormalizing to  $\infty$ . In this limit the two spins at sites 0 and 1 form a singlet and decouple from the rest of the spins which therefore correspond again to an open chain. Thus  $J' = J$  represents an unstable fixed point whereas  $J' = 0$  or  $\infty$  are stable fixed points. This conjecture was based on an analysis of the operator content at the uniform and open fixed points. This can be conveniently performed using non-Abelian bosonization [11]. The spin operators in the continuum limit are represented in terms of the fundamental field  $g_\beta^\alpha$  and currents  $\mathbf{J}_{L,R}$  of the  $SU(2)_1$  WZW model as

$$\mathbf{S}_j \approx (\mathbf{J}_L + \mathbf{J}_R) + \text{constant}(-1)^j \text{tr}(g\sigma). \quad (2.2)$$

By using the operator product expansion (OPE) one can show that

$$\mathbf{S}_j \cdot \mathbf{S}_{j+1} \approx \text{constant}(-1)^j \text{tr}g + \text{constant} \times \mathbf{J}_L \cdot \mathbf{J}_R. \quad (2.3)$$

Thus the modified link corresponds to a local interaction at the origin, in the low-energy effective Hamiltonian of the form:

$$\delta H \propto (J' - J) \text{tr}g(0). \quad (2.4)$$

Since  $g$  has scaling dimension  $1/2$ , this is relevant. (Recall that interactions occurring at only one point are relevant if they have dimension  $< 1$ .) To check the stability of the open fixed point we must consider its boundary operator content. Boundary operators are contained in the *chiral* part of the  $SU(2)$  WZW theory and, in this case, just correspond to the identity conformal tower. Thus the lowest-dimension operator corresponding to the spin at the end of an open chain is the current, of dimension 1. Coupling the two boundary spins together across the open link gives an operator of dimension 2 which is irrelevant. This conjecture is thus shown to be consistent with the stability of the open and uniform chain fixed points. The conjecture was further tested by numerical work [7]. The situation is quite different for two neighbouring modified links. In that case the uniform chain fixed point is stable and the open chain fixed point is unstable. The crucial difference at the uniform chain fixed point is that the relevant operator  $\text{tr}g$  cancels due to the  $(-1)^j$  factor in equation (2.3) leaving only the irrelevant operator  $d \text{tr}g/dx$  of dimension  $3/2$ . On the other hand, if we consider the limit  $J' \rightarrow 0$  on two links, we obtain two open chains and one decoupled spin. The RG equations in this case are the same as in the two-channel  $S = 1/2$  Kondo problem (the two channels corresponding to the left and right side of the impurity spin). An infinitesimal positive  $J'$  is

marginally relevant. In this case we think of the perturbed chain as ‘healing’. The effects of the local perturbation disappear in the low-energy effective Hamiltonian.

These two fixed points and the various RG flows between them can also be studied by the fusion technique [8], which played an essential role in the CFT study of the multi-channel Kondo problem. This gives a way of determining new boundary critical points from a starting reference critical point. In some cases this leads to the discovery of new critical points or a possible proof of the absence of additional critical points given certain completeness assumptions.

A starting point for fusion is to regard the right-movers as a second branch of left-movers. This is possible because left- and right-movers are, in a sense, decoupled in the conformal field theory; it is only the impurity interactions which couple them together. (We note that true boundary models, such as a spin chain on a semi-infinite line with interactions near the origin, can also be formulated entirely in terms of left-movers on the infinite line. However, in this case, no doubling of the number of degrees of freedom occurs.) In the spin chain problem we thus obtain a model with two flavours of left-moving WZW excitations,  $SU(2)_1 \times SU(2)_1$ , defined on an infinite line with the impurity interactions at the origin. It turns out [8] that to study the fixed points using fusion it appears necessary to then use a conformal embedding:

$$SU(2)_1 \times SU(2)_1 = SU(2)_2 \times \text{Ising}. \quad (2.5)$$

The  $SU(2)_2$  excitations carry the diagonal  $SU(2)$  quantum numbers and the Ising  $Z_2$  symmetry corresponds to switching the two  $SU(2)_1$  groups, or equivalently a parity transformation in the original formulation. The central charge adds up correctly, recalling that  $c = 1, 3/2$  and  $1/2$  for  $SU(2)_1, SU(2)_2$  and Ising respectively. The original non-chiral WZW fields at  $x = 0$  can now be represented as:

$$\mathbf{J}_L + \mathbf{J}_R = \mathbf{J} \quad \text{tr}(g\sigma) \propto \phi \quad \text{tr}g \propto \epsilon. \quad (2.6)$$

Here  $\mathbf{J}$  is the (chiral) current operator in  $SU(2)_2$ ,  $\phi$  is the spin-1 primary field of dimension  $1/2$  and  $\epsilon$  is the energy operator of the Ising model, also of dimension  $1/2$ . Note that these are all chiral operators and the dimensions of  $\epsilon$  and  $g$  are  $1/2$  of the dimensions of the corresponding scalar operators in the bulk Ising or WZW models (corresponding to the ‘left-moving parts’). In particular, in the case of the Ising model, we could think of  $\epsilon$  as corresponding to the chiral Majorana fermion field. The next step is to represent the various partition functions that occur with either open or uniform b.c.’s at both ends, for a finite system of length  $l$ , in terms of this conformal embedding. We think of the system as consisting of two sections of chain, both of length  $l$ , which may either be joined together, or separated at their two ends, at  $x = 0$  and  $l$ . Thus the uniform–uniform system is a periodic chain of length  $2l$ , the uniform–open system is a single open chain of length  $2l$  and the open–open system is two open chains, both of length  $l$ . We must also keep track of whether the number of microscopic  $S = 1/2$  operators is even or odd, giving a total of seven different partition functions. We express these partition functions, at temperature  $\beta^{-1}$ , in terms of the modular parameter

$$q \equiv e^{-\pi\beta/l}. \quad (2.7)$$

We write these partition functions in terms of the (chiral) characters  $\chi_j^{(1)}$  of the  $SU(2)_1$  model for  $j = 0$  or  $1/2$ ,  $\chi_j^{(2)}$  of the  $SU(2)_2$  model for  $j = 0, 1/2$  or  $1$ , and  $\chi_j^I$  of the Ising model for  $j = 0, 1/2$  and  $1$ . In the Ising case we have labelled the identity operator, order parameter and energy operator in terms of a parameter  $j = 0, 1/2$  and  $1$ , respectively. This is appropriate due to an isomorphism of the fusion rule coefficients and modular  $S$ -matrix between the  $SU(2)_2$  and Ising CFT’s. We also carefully take into account the universal  $1/l$  terms in the ground-state energy which are  $-\pi/12l$  for a periodic chain of length  $2l$  and  $-\pi/24l$  for a (single) open

chain of length  $l$ . We thus obtain the following partition functions, written first in terms of  $SU(2)_1 \times SU(2)_1$  characters and then in terms of  $SU(2)_2 \times$  Ising characters:

$$\begin{aligned}
 q^{1/12} Z_{\text{UU}}^e(q) &= \left[ \chi_0^{(1)}(q) \right]^2 + \left[ \chi_0^{(1/2)}(q) \right]^2 = \left[ \chi_0^{(2)}(q) + \chi_1^{(2)}(q) \right] \left[ \chi_0^1(q) + \chi_1^1(q) \right] \\
 q^{1/12} Z_{\text{UU}}^o(q) &= 2\chi_0^{(1)}(q)\chi_{1/2}^{(0)}(q) = 2\chi_{1/2}^{(2)}(q)\chi_{1/2}^1(q) \\
 q^{1/12} Z_{\text{UO}}^e(q) &= q^{1/16}\chi_0^{(1)}(\sqrt{q}) = \left[ \chi_0^{(2)}(q) + \chi_1^{(2)}(q) \right] \chi_{1/2}^1(q) \\
 q^{1/12} Z_{\text{UO}}^o(q) &= q^{1/16}\chi_{1/2}^{(1)}(\sqrt{q}) = \chi_{1/2}^{(2)}(q) \left[ \chi_0^1(q) + \chi_1^1(q) \right] \\
 q^{1/12} Z_{\text{OO}}^{\text{ee}}(q) &= \left[ \chi_0^{(1)}(q) \right]^2 = \chi_0^{(2)}(q)\chi_0^1(q) + \chi_1^{(2)}(q)\chi_1^1(q) \\
 q^{1/12} Z_{\text{OO}}^{\text{eo}}(q) &= \chi_{1/2}^{(1)}(q)\chi_0^{(1)}(q) = \chi_{1/2}^{(2)}(q)\chi_{1/2}^1(q) \\
 q^{1/12} Z_{\text{OO}}^{\text{oo}}(q) &= \left[ \chi_{1/2}^{(1)}(q) \right]^2 = \chi_0^{(2)}(q)\chi_1^1(q) + \chi_1^{(2)}(q)\chi_0^1(q)
 \end{aligned} \tag{2.8}$$

Here the superscripts denote even or odd length chains and the lower subscripts denote uniform or open b.c.'s. Now we use the fusion rules, which are isomorphic for  $SU(2)_2$  and Ising. These are

$$1/2 \times 1/2 = 0 + 1 \quad 1 \times 1/2 = 1/2 \quad 1 \times 1 = 0. \tag{2.9}$$

We can now check that fusion correctly takes us between the various partition functions in a way which corresponds to the various RG flows. For instance, suppose we start with  $Z_{\text{OO}}^{\text{ee}}$ , open–open b.c.'s with an even number of spins in each chain. Now consider adding one extra spin at  $x = 0$  which is weakly (and symmetrically) coupled to both chains. As discussed above, this induces a Kondo-type RG flow to the uniform–open fixed point, now with an odd number of sites. Since we have induced this flow by coupling to an  $S = 1/2$  impurity it is natural to associate this flow with fusion with  $S = 1/2$ . In fact, this is exactly what occurs in the Kondo problem. Now applying the fusion rules to  $Z_{\text{OO}}^{\text{ee}}$ , we see that both characters  $\chi_0^{(2)}$  and  $\chi_1^{(2)}$  get replaced by  $\chi_{1/2}^{(2)}$ , thus turning  $Z_{\text{OO}}^{\text{ee}}$  into  $Z_{\text{UO}}^o$ . Similarly fusion turns  $Z_{\text{OO}}^{\text{oo}}$  into  $Z_{\text{UO}}^o$ , and  $Z_{\text{OO}}^{\text{eo}}$  into  $Z_{\text{UO}}^e$ . We now add another spin at  $x = l$ , coupled to both chain ends and thus inducing a Kondo-type flow to the UU fixed points. Again it can be checked that fusion turns  $Z_{\text{UO}}^e$  into  $Z_{\text{UU}}^o$ , and  $Z_{\text{UO}}^o$  into  $Z_{\text{UU}}^e$ . It is also interesting to start with the OO case and consider fusion with the  $j = 1$  primary. In this case we see that  $Z_{\text{OO}}^{\text{ee}}$  is interchanged with  $Z_{\text{OO}}^{\text{oo}}$  while  $Z_{\text{OO}}^{\text{eo}}$  goes into itself. The associated RG flow now corresponds to introducing an  $S = 1$  impurity at  $x = 0$  or two  $S = 1/2$ 's with a ferromagnetic coupling. The flow back to the open fixed point corresponds to the  $S = 1$  impurity being screened by one  $S = 1/2$  spin from the end of each chain. Alternatively, if we have two weakly ferromagnetically coupled  $S = 1/2$  impurities, we may think of one of them attaching onto the end of each chain and asymptotically decoupling from each other. In either picture we end up getting the flow obtained from fusion. Similarly,  $Z_{\text{UO}}^e$  and  $Z_{\text{UO}}^o$  go into themselves under fusion with  $j = 1$ .

We may also consider fusion in the Ising sector. Note that the OO partition functions simply map into themselves under fusion with  $\epsilon$  but map into the OU partition functions under fusion with  $\sigma$ . The corresponding RG flows correspond to the two-impurity Kondo problem. We can imagine adding two  $S = 1/2$  impurities at  $x = 0$  which couple to each other with strength  $J''$  and also couple one to each chain with strength  $J'$ . The stable fixed points for this problem are just the open chain. If  $J''$  is too large compared to  $J'$ , the two impurities just form a singlet and decouple. If  $J''$  is too small, then one impurity can couple onto the end of each chain, the chains remaining open. However, if the ratio of  $J''$  to  $J'$  is just right, then the system can heal, flowing to the uniform fixed point. These three cases correspond, respectively, to the inter-impurity singlet, Kondo screened and non-Fermi liquid fixed points in the two-impurity Kondo problem. The second case (independent impurity screening by the chains) corresponds

to fusion with  $\epsilon$  while the uniform chain corresponds to fusion with  $\sigma$ . Again, the same type of fusion was used in the CFT treatment of the two-impurity Kondo problem. We also note that the same fusion process describes the effect of adding two more impurities at  $x = l$  to the UO chain. Fusion with  $\epsilon$  corresponds to attaching one spin to each open chain but fusion with  $\sigma$  corresponds to the defect healing, producing a flow to the uniform fixed point.

The Ising symmetry may be given a physical interpretation. The  $Z_2$  symmetry corresponds to parity, reflection around the origin. One way of seeing this is to note that the boundary operator introduced by a single modified link,  $\text{tr}g(0)$ , corresponds to  $\epsilon$  after our conformal embedding and this operator corresponds to a boundary magnetic field in the Ising model [12]. A single modified link breaks this symmetry explicitly, whereas two equally modified links (between  $(-1)$  and  $0$  and between  $0$  and  $1$ ) do not. The analogue of Ising order in the  $S = 1/2$  chain is a spontaneously dimerized state. This does not occur for the Heisenberg model, although it does with sufficiently strong next-nearest-neighbour interaction, as in the Majumdar–Ghosh model [9]. The quasi-long range dimer–dimer correlation function indicates that the Heisenberg model is in a critical state with respect to this type of order. Strengthening the coupling on a single link locally favours one of the two dimer states. It is like applying a boundary magnetic field to the critical Ising model. This is a relevant perturbation and in the infrared limit is like applying a spin-up boundary condition. Thus the flow from uniform to open fixed points in the spin chain corresponds to the flow from free to fixed boundary conditions in the Ising model. In both cases the model is responding to a symmetry-breaking perturbation acting only at the boundary.

The fact that no new partition functions are obtained by fusion with all primaries, starting from OO b.c.'s, lends support to the conjecture that only open and uniform fixed points occur in this problem. Conversely, the fact that U can be obtained from O supports the general notion that fusion provides a complete set of fixed points starting from a suitable reference fixed point. However, it must be admitted that the fusion construction has not added very much to our understanding of the  $SU(2)$  spin chain. The two basic fixed points are both trivial and could be obtained by elementary methods. As we shall see in the next section, the situation is quite different in the  $SU(3)$  case. Now non-trivial fixed points occur which cannot be obtained by elementary methods. Fusion provides a powerful method of solving for the properties of these fixed points. Furthermore, it seems reasonable to conjecture that the set of fixed points obtained by fusion may give the complete set of conformally invariant boundary conditions for the model.

### 3. $SU(3)$ case

The Hamiltonian for the  $SU(3)$  ‘spin’ chain may be written as in equation (1.2) with the constraint of equation (1.1). Alternatively, we may introduce generators of  $SU(3)$ ,  $T^A$ , with  $A = 1, 2, 3, \dots, 8$ , a complete set of traceless Hermitian matrices normalized so

$$\text{tr}T^A T^B = (1/2)\delta^{AB} \quad (3.1)$$

and introduce associated operators

$$S_j^A \equiv \eta_j^{\dagger\alpha} (T^A)_{\alpha\beta} \eta_{j\beta}. \quad (3.2)$$

The Hamiltonian may then be written as

$$H = \sum_{j^A} J_j S_j^A S_{j+1}^A. \quad (3.3)$$

The continuum limit can be derived, for example, using a weak-coupling Hubbard model representation and then extrapolating to an infinite Hubbard coupling constant [10]. We thus

keep only Fourier modes of the fermion fields  $\eta^\alpha$  near the Fermi points  $k_F = \pm\pi/3$ , introducing left- and right-movers:

$$\eta_j^\alpha \approx e^{i\pi j/3} \eta_L^\alpha(j) + e^{-i\pi j/3} \eta_R^\alpha(j). \quad (3.4)$$

The resulting interacting fermion model can be treated using non-Abelian bosonization. We thus introduce an  $SU(3)_1$  WZW nonlinear  $\sigma$  model field  $g_\beta^\alpha$  to represent the spin degrees of freedom and an additional charge boson field. The charge boson develops a gap from the Hubbard interaction and can be dropped from the Hamiltonian which is then just the conformally invariant WZW model up to irrelevant interactions (and before including impurity effects). The original spin operators are then represented at low energies as

$$S_j^A \approx (J_L^A + J_R^A)(j) + [\text{constant} \times e^{i2\pi j/3} \text{tr}(g(j)T^A) + \text{h.c.}] \quad (3.5)$$

where  $J_{L/R}^A$  are the  $SU(3)$  currents

$$J_{L,R}^A \equiv \eta_{L/R}^{\dagger\alpha} (T^A)_\alpha^\beta \eta_{\beta L,R}. \quad (3.6)$$

$g$  has a scaling dimension of  $2/3$ . Using the OPE it can be shown that

$$\sum_A S_j^A S_{j+1}^A \approx [e^{i2\pi j/3} \times \text{constant} \times \text{tr} g + \text{h.c.}] + \text{constant} \times \sum_A J_L^A J_R^A. \quad (3.7)$$

The  $2k_F$  part has dimension  $2/3$ . We now consider a single modified exchange coupling  $J_0 = J'$  between sites 0 and 1. This introduces the interaction in the low-energy effective Hamiltonian:

$$\delta H \propto (J' - J) \text{tr} g + \text{h.c.} \quad (3.8)$$

Since this has dimension  $2/3 < 1$ , it is relevant. We now consider the possible infrared fixed point of the RG flow. In the case  $J' < J$  it is plausible that  $J'$  simply renormalizes to 0 as in the  $SU(2)$  case corresponding to an open chain fixed point. However, the situation is now quite different for  $J' > J$ . Unlike the  $SU(2)$  case,  $J' \rightarrow \infty$  is *not* a stable fixed point. In the  $SU(3)$  case, if we take  $J' \rightarrow \infty$  we project the two ‘spins’ at sites 0 and 1 into the  $\bar{3}$  representation, rather than into a singlet, as for  $SU(2)$ . Even at  $J' \rightarrow \infty$  a residual interaction of  $O(J)$  exists between this effective  $\bar{3}$  spin and the neighbouring spins at sites  $(-1)$  and  $2$ .

Since the sign of this residual interaction is important, we calculate it explicitly. This is most conveniently done in terms of the spin operators:

$$S_\beta^\alpha \equiv \eta^{\dagger\alpha} \eta_\beta - (1/3) \delta_\beta^\alpha. \quad (3.9)$$

This acts on the three-representation state,

$$|\alpha\rangle \equiv \eta^{\dagger\alpha} |0\rangle \quad (3.10)$$

as

$$S_\beta^\alpha |\gamma\rangle = \delta_\beta^\gamma |\alpha\rangle - (1/3) \delta_\beta^\alpha |\gamma\rangle. \quad (3.11)$$

The  $\bar{3}$  state (on a single site) corresponds to two fermions:

$$|\alpha, \beta\rangle = -|\beta, \alpha\rangle = \eta^{\dagger\alpha} \eta^{\dagger\beta} |0\rangle. \quad (3.12)$$

The projected  $\bar{3}$  state on sites 0 and 1, obtained at  $J' \rightarrow \infty$  can be written as

$$|\alpha, \beta\rangle_{01} \equiv |\alpha\rangle_0 \times |\beta\rangle_1 - |\beta\rangle_0 \times |\alpha\rangle_1 \quad (3.13)$$

where the first and second factor refer to sites 0 and 1 respectively. Now consider the action of  $S_{0\beta}^\alpha$  on this  $\bar{3}$  state:

$$S_{0\beta}^\alpha |\gamma, \delta\rangle_{01} = \delta_\beta^\gamma |\alpha\rangle_0 \times |\delta\rangle_1 - \delta_\beta^\delta |\alpha\rangle_0 \times |\gamma\rangle_1 - (1/3) \delta_\beta^\alpha |\gamma, \delta\rangle_{01}. \quad (3.14)$$

Finally we project this back into the low-energy  $\bar{3}$  subspace, by antisymmetrizing, giving

$$P S_{0\beta}^\alpha |\gamma, \delta\rangle_{01} = (1/2) [\delta_\beta^\gamma |\alpha, \delta\rangle_{01} - \delta_\beta^\delta |\alpha, \gamma\rangle_{01}] - (1/3) \delta_\beta^\alpha |\gamma, \delta\rangle_{01}. \quad (3.15)$$



Thus we see that, upon projecting into the low-energy subspace of the  $\bar{3}$  representation,

$$\mathcal{P}S_{0,\beta}^\alpha\mathcal{P} = (1/2)S_{\text{eff},\beta}^\alpha - (1/6)\delta_\beta^\alpha. \tag{3.16}$$

Therefore, the residual exchange interaction between site  $(-1)$  and the effective  $\bar{3}$  spin has the value  $J/2 > 0$ . The case of a small positive coupling to the effective  $\bar{3}$  impurity spin gives a Kondo type RG equation and is hence marginally relevant. Thus, it appears that  $J'$  *does not* renormalize to  $\infty$  when  $J' > J$ . It is therefore reasonable to expect that some sort of non-trivial fixed point occurs in this problem. We construct this fixed point explicitly below using the fusion method.

First, however, we consider the case of two modified (but equal) exchange couplings on neighbouring links  $0-1$  and  $1-2$ . A difference immediately appears with the  $SU(2)$  case. The relevant operator,  $\text{constant} \times \text{tr}g + \text{h.c.}$  appears in the  $SU(3)$  case because the oscillating factors  $e^{i2\pi j/3}$  do not cancel between two neighbouring links. Thus modifying two neighbouring links is also a relevant perturbation. We may consider the possible RG flows by again considering the limit where  $J' \rightarrow \infty$  or  $0$ . Note that when  $J' \rightarrow \infty$ , three neighbouring sites form an  $SU(3)$  singlet:

$$\epsilon_{\alpha\beta\gamma}|\alpha\rangle_0 \times |\beta\rangle_1|\gamma\rangle_2. \tag{3.17}$$

This effectively breaks the chain into two disconnected pieces, corresponding to the open fixed point. Since this is a stable fixed point, it is plausible that it any  $J' > J$  flows to it. On the other hand, when  $J' \rightarrow 0$ , we get two chains with a Kondo coupling to an impurity in the  $3$  representation. This corresponds to the two-channel  $SU(3)$  Kondo model, as can be seen from the non-Abelian bosonization of this model [13]. The two channels correspond to the decoupled chains on the two sides of the impurity. This Kondo interaction is marginally relevant, so  $J' = 0$  is *not* a stable fixed point. Thus it appears that there must be a non-trivial fixed point with two modified links in the case  $J' < J$ .

We now wish to study this problem using the fusion method. To do this we must first introduce an appropriate conformal embedding. Following the  $SU(2)$  case, we regard the right-movers as a second branch of left-movers and then introduce an  $SU(3)_2$  WZW model representing the diagonal  $SU(3)$  degrees of freedom. We then must introduce another CFT representing the coset  $SU(3)_1 \times SU(3)_1/SU(3)_2$ . This turns out to be the three-state Potts model:

$$SU(3)_1 \times SU(3)_2 = SU(3)_2 \times \text{Potts}. \tag{3.18}$$

The conformal charges add up correctly:  $2 + 2 = 16/5 + 4/5$ . The  $SU(3)_2$  WZW model has primary fields in the  $3$ ,  $6$  and  $8$  representations with scaling dimensions  $4/15$ ,  $2/3$  and  $3/5$ , respectively (as well as the conjugate fields in the  $\bar{3}$  and  $\bar{6}$  representations). The Potts model contains two conjugate pairs of fields,  $\sigma, \sigma^\dagger$  of dimension  $1/15$  and  $\psi, \psi^\dagger$  of dimension  $2/3$  as well as the Hermitian field  $\epsilon$  of dimension  $2/5$ . Note that we only obtain the ‘chiral factors’ of the various primary operators and these scaling dimensions are all for these chiral factors. The lattice ‘spin’ operators are then represented in terms of these degrees of freedom as

$$S_j^A \approx J^A + \phi^A [e^{i2\pi j/3} \times \text{constant} \times \sigma + \text{h.c.}]. \tag{3.19}$$

Here  $\phi^A$  is the  $SU(3)_2$  adjoint representation field. The  $2k_F$  part of the spin produced is

$$\sum_A S_j^A S_{j+1}^A \approx [e^{i2\pi j/3} \times \text{constant} \times \psi + \text{h.c.}]. \tag{3.20}$$

We can again write down the partition functions corresponding to trivial uniform or open boundary conditions, more or less by inspection. In this case we get a different result depending on the length of the chains mod 3, represented by superscripts 0, 1 or 2. We find that it is

necessary to introduce additional characters in the Potts sector which do not occur in the bulk Potts spectrum but are legitimate conformal towers occurring in the bulk spectrum of the other  $c = 4/5$  CFT, the tetra-critical Ising model. This phenomena was already encountered in our discussion of boundary critical points in the Potts model. Characters not appearing in the bulk spectrum can occur in the spectrum with boundaries (in the open string channel only) and can be used in constructing boundary conditions by fusion. These additional characters correspond to primary fields of dimension  $1/8, 13/8, 1/40$  and  $21/40$ . The partition functions corresponding to the various uniform or open b.c.’s can then be written as follows, first in terms of  $SU(3)_1 \times SU(3)_1$  characters and then in terms of  $SU(3)_2 \times$  Potts characters:

$$\begin{aligned}
q^{1/6}Z_{\text{UU}}^0(q) &= [\chi_1^{(1)}(q)]^2 + [\chi_3^{(1)}(q)]^2 + [\chi_{\bar{3}}^{(1)}(q)]^2 \\
&= \chi_1^{(2)}(q) [\chi_I^P(q) + \chi_\psi^P(q) + \chi_{\psi^\dagger}^P(q)] + \chi_8^{(2)}(q) [\chi_\epsilon^P(q) + \chi_\sigma^P(q) + \chi_{\sigma^\dagger}^P(q)] \\
q^{1/6}Z_{\text{UU}}^1(q) &= 2\chi_1^{(1)}(q)\chi_3^{(1)}(q) + [\chi_{\bar{3}}^{(1)}(q)]^2 \\
&= \chi_3^{(2)}(q) [\chi_\epsilon^P(q) + \chi_\sigma^P(q) + \chi_{\sigma^\dagger}^P(q)] + \chi_6^{(2)}(q) [\chi_I^P(q) + \chi_\psi^P(q) + \chi_{\psi^\dagger}^P(q)] \\
q^{1/6}Z_{\text{UU}}^2(q) &= 2\chi_1^{(1)}(q)\chi_{\bar{3}}^{(1)}(q) + [\chi_3^{(1)}(q)]^2 \\
&= \chi_3^{(2)}(q) [\chi_\epsilon^P(q) + \chi_\sigma^P(q) + \chi_{\sigma^\dagger}^P(q)] + \chi_6^{(2)}(q) [\chi_I^P(q) + \chi_\psi^P(q) + \chi_{\psi^\dagger}^P(q)] \\
q^{1/6}Z_{\text{UO}}^0(q) &= q^{1/8}\chi_1^{(1)}(\sqrt{q}) = \chi_1^{(2)}(q)\chi_{1/8}^P(q) + \chi_8^{(2)}(q)\chi_{1/40}^P(q) \\
q^{1/6}Z_{\text{UO}}^1(q) &= q^{1/8}\chi_3^{(1)}(\sqrt{q}) = \chi_3^{(2)}(q)\chi_{1/40}^P(q) + \chi_6^{(2)}(q)\chi_{1/8}^P(q) \\
q^{1/6}Z_{\text{UO}}^2(q) &= q^{1/8}\chi_{\bar{3}}^{(1)}(\sqrt{q}) = \chi_3^{(2)}(q)\chi_{1/40}^P(q) + \chi_6^{(2)}(q)\chi_{1/8}^P(q) \\
q^{1/6}Z_{\text{OO}}^{00}(q) &= [\chi_1^{(1)}(q)]^2 = \chi_1^{(2)}(q)\chi_I^P(q) + \chi_8^{(2)}(q)\chi_\epsilon^P(q) \\
q^{1/6}Z_{\text{OO}}^{01}(q) &= \chi_1^{(1)}(q)\chi_3^{(1)}(q) = \chi_3^{(2)}(q)\chi_\sigma^P(q) + \chi_6^{(2)}(q)\chi_\psi^P(q) \\
q^{1/6}Z_{\text{OO}}^{02}(q) &= \chi_1^{(1)}(q)\chi_{\bar{3}}^{(1)}(q) = \chi_3^{(2)}(q)\chi_{\sigma^\dagger}^P(q) + \chi_6^{(2)}(q)\chi_{\psi^\dagger}^P(q) \\
q^{1/6}Z_{\text{OO}}^{11}(q) &= [\chi_3^{(1)}(q)]^2 = \chi_6^{(2)}(q)\chi_I^P(q) + \chi_3^{(2)}(q)\chi_\epsilon^P(q) \\
q^{1/6}Z_{\text{OO}}^{12}(q) &= \chi_3^{(1)}(q)\chi_{\bar{3}}^{(1)}(q) = \chi_1^{(2)}(q)\chi_\psi^P(q) + \chi_8^{(2)}(q)\chi_\sigma^P(q) \\
q^{1/6}Z_{\text{OO}}^{22}(q) &= [\chi_{\bar{3}}^{(1)}(q)]^2 = \chi_6^{(2)}(q)\chi_I^P(q) + \chi_3^{(2)}(q)\chi_\epsilon^P(q).
\end{aligned} \tag{3.21}$$

Although we do not know formal proofs of these identities, we have checked them using MATHEMATICA up to the level  $q^{40}$  in the expansion in  $q$ , in all cases.

Following our treatment of the  $SU(2)$  case in the previous section, we now begin with the open–open boundary conditions and consider the effect of all possible fusion processes in either  $SU(3)_2$  or Potts sectors. The fusion rules for  $SU(3)_2$  are

$$\begin{aligned}
3 \times 3 &\rightarrow \bar{3} + 6 & 3 \times \bar{3} &\rightarrow 1 + 8 & 3 \times 6 &\rightarrow 8 \\
3 \times \bar{6} &\rightarrow \bar{3} & 3 \times 8 &\rightarrow 3 + \bar{6} & 6 \times 6 &\rightarrow \bar{6} \\
6 \times 8 &\rightarrow \bar{3} & 6 \times \bar{6} &\rightarrow 1 & 8 \times 8 &\rightarrow 1 + 8.
\end{aligned} \tag{3.22}$$

The fusion rules and modular S-matrix in the  $W$ -invariant sector of the Potts model are equivalent to those of  $SU(3)_2$  with the identification:

$$\begin{aligned}
1 &\rightarrow I & 8 &\rightarrow \epsilon & 3 &\rightarrow \sigma \\
\bar{3} &\rightarrow \sigma^\dagger & 6 &\rightarrow \psi & \bar{6} &\rightarrow \psi^\dagger.
\end{aligned} \tag{3.23}$$

The fusion rules and modular  $S$ -matrix for the extended Potts algebra are given in tables 1 and 2 of [5]. We start with  $Z_{OO}^{00}$ , although we do not expect to obtain a different result if we begin with the other possible  $Z_{OO}^{ij}$  cases. Let us first consider fusion in the  $SU(3)_2$  sector. We see that fusion with 3 (or equivalently  $\bar{3}$ ) or 8 gives a new partition function, not in the list in equation (3.21). On the other hand, fusion with 6 (or equivalently  $\bar{6}$ ) gives  $Z_{OO}^{11}$  (or  $Z_{OO}^{22}$ ). These results are more or less what we should have expected based on the above discussion of the RG flows. Fusion with 3 should correspond to weakly coupling one new impurity spin to both chains at the origin. This is related to weakening two neighbouring links, in the case of antiferromagnetic coupling, which was argued above to lead to a non-trivial fixed point. Fusion with  $\bar{3}$  would correspond to adding two impurity spins between the ends of the open chains at the origin, with the two spins strongly coupled together antiferromagnetically in order to obtain the  $\bar{3}$  representation. This is related to the case of one strengthened link which should also lead to a non-trivial fixed point. Fusion with 6 is related to adding two impurity spins which are ferromagnetically coupled to each other. This is related to weakening the (initially antiferromagnetic) coupling between two spins which was argued to lead to a trivial fixed point, the open chain. The case of fusion with 8 is less obviously related to the previous discussion. It is instructive, at this point, to consider a second fusion with the conjugate operator, corresponding to the same process taking place at  $x = l$ . Fusion with 6 then  $\bar{6}$  produces the open–open fixed point, as expected. On the other hand, fusion with 3 then  $\bar{3}$  or 8 then 8 leads to the same new partition function, which corresponds to having the non-trivial b.c. at both ends of the system. The fact that the same partition function results from either double fusion process indicates that there is only one new fixed point occurring, not two. Next we consider fusion in the Potts sector. We expect that this corresponds to adding three impurity spins with various types of self-couplings. We find that fusion with  $\sigma$  (or equivalently  $\sigma^\dagger$ ) or  $\epsilon$  leads to a non-trivial fixed point. This appears to be the same one obtained from  $SU(3)_2$  fusion, as seen by checking the result of double fusion. On the other hand, fusion with  $\psi$  leads to the  $Z_{OO}^{12}$  partition function, indicating that we simply obtain a flow to open–open boundary conditions. This seems to correspond to three impurity spins coupled asymmetrically; one attaches to one chain and two attach to the other. Finally, we may consider fusion with the extended Potts operators. We find that fusion with the dimension  $1/8$  or  $13/8$  operators gives  $Z_{OO}^0$ . This corresponds to coupling three impurity spins between the ends of the open chains and obtaining the uniform fixed point. (This is an unstable fixed point for this process.) Finally, fusion with the operators of dimensions  $1/40$  or  $21/40$  gives another new fixed point, *not* equivalent to the one discussed earlier.

The first non-trivial fixed point, discussed above, can be given an interpretation related to the Potts model. In the Potts model there is a ‘mixed’ fixed point corresponding to the Potts variables on the boundary fluctuating back and forth between two of the three possible states. We may interpret these three Potts states as corresponding to the three possible trimerization patterns of the  $SU(3)$  spin chain, i.e. the trimers form on sites  $3j - (3j + 1) - (3j + 2)$  or  $(3j - 1) - 3j - (3j + 1)$  or  $(3j - 2) - (3j - 1) - 3j$  for all integer  $j$ . Note that two strengthened neighbouring links on links 0–1 and 1–2 favour trimer formation on 0–1–2, corresponding to a fixed b.c. in the Potts model and an open b.c. in the spin chain. Similarly one weakened bond on 0–1 favours the 1–2–3 trimerization pattern, again corresponding to the open b.c. However, one strengthened bond on 0–1 equally favours two trimerization patterns, 0–1–2 or  $(-1) - 0 - 1$ . We may think of the non-trivial fixed point as being one in which the trimers resonate between these two states near the origin. This is very analogous to the mixed fixed point in the Potts model. Hence it is appropriate to refer to this state in the  $SU(3)$  chain as the mixed fixed point. Similarly, weakening two bonds on 0–1 and 1–2 equally favours two trimerization patterns 1–2–3 or  $(-1) - 0 - 1$ . Again this gives the mixed fixed point. We

note that fusion in the Potts sector of the  $SU(3)$  chain connects the fixed points in a way which corresponds to that in the Potts model. Starting from a fixed b.c. in the Potts model, fusion with  $\psi$  (or  $\psi^\dagger$ ) gives the other fixed b.c.’s but fusion with  $\sigma$ ,  $\sigma^\dagger$  or  $\epsilon$  gives the mixed b.c. Fusion with the  $1/8$  operator gives the free b.c. in the Potts model corresponding to the uniform b.c. in the  $SU(3)$  chain. Fusion with the  $1/40$  operator in the Potts model gives a new fixed point first discussed in [5]. This corresponds to the new fixed point in the  $SU(3)$  chain. So far, we have been unable to understand what sort of microscopic impurity couplings in the  $SU(3)$  chain would produce a flow to this new fixed point. A related difficulty is that the  $Z_3$  symmetry in the impurity problems may only be defined in the low-energy continuum limit. While this symmetry can be identified as translations by 0, 1 or 2 sites in the uniform chain, this translational symmetry is always broken in the impurity models. Note that we were able to circumvent the analogous problem in the case of the  $SU(2)$  chain with an impurity by identifying the  $Z_2$  symmetry with reflection about a site, rather than translation by one site. Both these symmetry operations have the effect of interchanging the two dimerized ground states. On the other hand, in the  $SU(3)$  case there appear to be no analogous symmetries which interchange the trimerized ground states and which remain symmetries with impurity interactions present.

It is instructive to consider the boundary operator content at the mixed fixed point, obtained by double fusion. This is

$$1 \times I + 8 \times I + (2 \times) 8 \times \epsilon + 1 \times \epsilon. \quad (3.24)$$

Here the first and second factors correspond to operators in the  $SU(3)_2$  and Potts sector, respectively, and the factor of 2 indicates that two such operators occur. We see that there is only one  $SU(3)$  symmetric relevant operator,  $1 \times \epsilon$ . This has a natural interpretation related to the above discussion. The mixed fixed point corresponds to resonance between two different trimerization patterns near the origin. Modifying exchange couplings so as to favour one of these over the other is a relevant perturbation. For instance, if we obtain the mixed fixed point by strengthening the coupling on link 0–1, then strengthening the coupling on 1–2 is relevant since it then favours the 0–1–2 trimerization over (–1)–0–1. In fact, the corresponding relevant operator also occurs at the mixed fixed point in the Potts model.

#### 4. Connection with the $tJV$ model

The  $SU(3)$  ‘spin’ chain is equivalent to the  $tJV$  model for a special choice of the parameters  $J$  and  $V$  [6]. The Hamiltonian is

$$H = \sum_j \left\{ [-tP\psi_j^{\dagger a}\psi_{j+1,a}P + \text{h.c.}] + JS_j \cdot S_{j+1} + Vn_jn_{j+1} \right\}. \quad (4.1)$$

Here  $P$  projects out states with no double occupancy, and  $S_j$  and  $n_j$  are the electron spin and charge operators on site  $j$ :

$$\begin{aligned} S_j &\equiv \psi_j^{\dagger a} \frac{\sigma_a^b}{2} \psi_{jb} \\ n_j &\equiv \psi_j^{\dagger a} \psi_{ja}. \end{aligned} \quad (4.2)$$

The spin indices,  $a, b$ , represented by Latin letters, are summed from 1 to 2 only. Greek letters are used for the  $SU(3)$  indices summed over 0, 1, 2. This model is equivalent to the  $SU(3)$  spin chain for  $J = 2t$ ,  $V = 3t/2$ .

We remark that this is *not* the same as the supersymmetric  $tJ$  model which has  $J = 2t$  but  $V = -t/2 = -J/4$ . (The latter is referred to as a  $tJ$  model rather than a  $tJV$  model because the  $tJ$  model is sometimes written as in equation (4.1) with  $V = -J/4$ .)

The equivalence of the  $SU(3)$  spin chain with this  $tJV$  model is established by writing the  $SU(3)$  spin chain Hamiltonian in the form

$$H = \frac{1}{2}J \sum_j S_{j\beta}^\alpha S_{j\alpha}^\beta \quad (4.3)$$

with

$$S_{j\beta}^\alpha \equiv \eta_j^{\dagger\alpha} \eta_{j\beta} \quad (4.4)$$

and then mapping into  $tJV$  operators as follows:

$$\begin{aligned} \eta_j^{\dagger a} \eta_{jb} &\rightarrow \psi_j^{\dagger a} \psi_{jb} \\ \eta_j^{\dagger 0} \eta_{j0} &= 1 - \eta_j^{\dagger a} \eta_{ja} \rightarrow 1 - \psi_j^{\dagger a} \psi_{ja} \equiv 1 - n_j \\ \eta_j^{\dagger a} \eta_{j0} &\rightarrow \exp\left[i\pi \sum_{l=1}^{j-1} n_l\right] \psi_j^{\dagger a} (1 - n_{j\check{a}}). \end{aligned} \quad (4.5)$$

Here  $\check{a}$  denotes the other index, i.e.  $\check{1} = 2$  and  $\check{2} = 1$ . Note the familiar Jordan–Wigner string operator in the last line of equation (4.5) which turns the right-hand side into a commuting (bosonic) object. It may be verified that the mapping of equation (4.5) respects the  $SU(3)$  commutation relations:

$$[S_{j\beta}^\alpha, S_{k\epsilon}^\gamma] = \delta_{jk}[\delta_\beta^\gamma S_{j\epsilon}^\alpha - \delta_\epsilon^\alpha S_{j\beta}^\gamma]. \quad (4.6)$$

Using equation (4.5), it is straightforward to check that the Hamiltonian of equation (4.3) maps into the  $tJV$  Hamiltonian with  $t = 1$ ,  $J = 2$  and  $V = 3/2$ . If the  $SU(3)$  Hamiltonian is written with periodic b.c.'s then we obtain the  $tJV$  model with b.c.'s that are periodic when the total number of electrons is even but anti-periodic when the total number is odd. Furthermore, the density of electrons must be fixed at  $2/3$  (and the total spin at 0) to correspond to the  $SU(3)$ -invariant ground state.

It is also interesting to consider the continuum limit of the  $SU(3)$  spin chain using Abelian bosonization, rather than the non-Abelian bosonization used in the previous section. This amounts to representing the  $SU(3)_1$  WZW model in terms of two free bosons, a correspondence which is consistent with the central charge  $c = 2$ . The two bosons can then be identified with the charge and spin bosons that are familiar in the continuum limit of the Hubbard or  $tJV$  models. We can then determine the compactification radii of the charge boson as well as that of the spin boson at the  $SU(3)$ -invariant point. (The result for the spin boson is the well-known value corresponding to  $SU(2)$  invariance.) At these special radii, the continuum two-boson model becomes equivalent to the  $SU(3)_1$  WZW model.

The Abelian bosonization of the continuum limit field theory for the  $\eta$  fermions introduces three boson,  $\phi_\alpha$  for the three fermion fields. These may be rewritten in terms of a more convenient basis:  $\phi$ ,  $\phi_c$  and  $\phi_s$ , defined as:

$$\begin{aligned} \phi &\equiv \frac{\phi_1 + \phi_2 + \phi_0}{\sqrt{3}} \\ \phi_c &\equiv \frac{\phi_1 + \phi_2 - 2\phi_0}{\sqrt{6}} \\ \phi_s &\equiv \frac{\phi_1 - \phi_2}{\sqrt{2}}. \end{aligned} \quad (4.7)$$

$\phi$  represents the pseudo-charge boson in the three-component  $\eta$  field theory. The Hubbard interaction in the  $\eta$  theory produces a gap for  $\phi$  which may be dropped from the Hamiltonian.  $\phi_c$  is the charge boson in the two-component  $\psi$  field theory and  $\phi_s$  is the  $SU(2)$  spin boson.

To determine the radius of the charge boson we may write down the resulting bosonized form for various operators and compare to the standard result. For instance,

$$\psi_{L1}^\dagger \psi_{R1} \propto e^{i\sqrt{4\pi}\phi_1} \rightarrow e^{i[\sqrt{4\pi/3}\phi + \sqrt{2\pi/3}\phi_c + \sqrt{2\pi}\phi_s]} \quad (4.8)$$

We expect the Hubbard interaction to produce a non-zero expectation value:

$$\langle e^{i\sqrt{4\pi/3}\phi} \rangle \neq 0 \quad (4.9)$$

so we may drop the first term in the exponent in equation (4.8). On the other hand, if we had started from the  $\psi$  fermion model (the ordinary two-component Hubbard model) we would have written

$$\psi_{L1}^\dagger \psi_{R1} \propto e^{i[\phi_c/R_c + \phi_s/R_s]} \quad (4.10)$$

In the non-interacting limit  $R_c = R_s = 1/\sqrt{2\pi}$ .  $SU(2)$ -invariant interactions renormalize  $R_c$ , but not  $R_s$ . Comparing equation (4.10) to (4.8) we see that the  $SU(3)_1$  WZW model, the continuum limit of the  $SU(3)$ -invariant  $tJV$  model, has

$$R_c = \sqrt{3/2\pi}. \quad (4.11)$$

In the notation of Kane and Fisher, this corresponds to

$$g_\sigma = 2 \quad g_\rho = 2/3. \quad (4.12)$$

At this value of  $R_c$ ,  $\psi_{L1}^\dagger \psi_{R1}$  has scaling dimension  $2/3$ , corresponding to the 11 component of the  $SU(3)_1$  WZW field,  $g_{11}$ . Here we have used the fact that the  $SU(3)$  symmetry protects the radius of the charge (as well as spin) boson from renormalizing as the Hubbard interaction is increased to  $\infty$ . Thus we see that the spinful Luttinger liquid model at the special values of  $g_\rho$  and  $g_\sigma$  where it was solved exactly by Yi and Kane [3] has a hidden  $SU(3)$  symmetry. This provides some understanding of the solvability at this special point and suggests that the  $SU(3)$  approach used here is the most natural way of studying the problem.

## 5. $SU(3)$ symmetry breaking

A natural question to ask, at this point, is what happens if we allow boundary interactions that break the  $SU(3)$  symmetry down to  $SU(2) \times U(1)$ ? This would correspond to starting with the  $SU(3)$ -invariant  $tJV$  model in the bulk but then allowing arbitrary strength hopping, spin exchange and Coulomb repulsion on the modified links, for example. Alternatively, we could apply an  $SU(3)$  ‘field’ which favours holes over electrons at one or more sites. Referring to equation (3.24), we see that in addition to the  $SU(3)$ -invariant relevant operator at the mixed fixed point there is also one relevant operator transforming under the adjoint representation of  $SU(3)$ . This contains one  $SU(2) \times U(1)$  singlet field, the 3–3 component of the adjoint field,  $\phi_{33}$ . There are also two marginal operators with the same  $SU(3)$  transformation properties. We expect that these correspond to interactions that break both parity and  $SU(3)$ . This follows from assigning an odd parity quantum number to the Potts operator  $\epsilon$  which appears as a factor in these marginal operators. It is interesting to consider what happens in the case of two weakened links, on  $(-1)-0$  and  $0-1$ , if we then apply a ‘field’ (i.e. local potential) at the origin, thus respecting the parity symmetry. It seems plausible that this is a relevant perturbation and generates the operator  $\phi_{33}$ . It is clear that a large local potential at the origin will lead to a trivial fixed point. A positive potential, favouring a hole at the origin, gives the open fixed point, corresponding to zero conductance for spin and charge. On the other hand, a negative potential, favouring one electron at the origin, produces a trivial but non- $SU(3)$ -invariant fixed point, as argued by Kane and Fisher [2]. The single electron at the origin acts as a Kondo

impurity. Since  $g_\rho < 1$ , it blocks charge transport through the origin but allows spin transport. This corresponds to a phase with perfect transmission for spin but perfect reflection for charge. The non-trivial, mixed, critical point is sitting at a resonance, in between these two stable fixed points. Kane and Fisher identified their non-trivial resonant fixed point as an unstable fixed point separating precisely these two stable phases. Remarkably,  $SU(3)$  symmetry (at the boundary as well as in the bulk) puts the system exactly on resonance.

We obtain an analogous result by considering the case of one strengthened link with breaking of  $SU(3)$  symmetry. It is instructive to consider the effect of the  $SU(3)$  symmetry breaking in the limit where  $J' \rightarrow \infty$ . Then, the  $\bar{3}$  representation is projected out on the sites 0–1. The three states of the  $\bar{3}$  representation consist of an  $SU(2)$  doublet, with one electron (of either spin) hopping back and forth between sites 0 and 1 in a zero momentum state, and of an  $SU(2)$  singlet state with one electron on site 0 and one electron on site 1. In other words, we may either increase  $t$  to favour the doublet state or increase  $J$  to favour the singlet.  $SU(3)$  symmetry breaking favours either the doublet or the singlet. In the case where the singlet is favoured we again get the open fixed point since there is zero charge or spin transport through sites 0–1 in this case. However, when the doublet is favoured the single electron shared by sites 0 and 1 again acts like a Kondo impurity, allowing spin transport but not charge transport. Again we expect flow to a fixed point with perfect transmission for spin but perfect reflection for charge.

We also consider breaking of the  $SU(3)$  symmetry down to  $SU(2) \times U(1)$  in the bulk. This corresponds to the  $tJV$  model with general parameters and chemical potential. The relevant boundary operator,  $\phi_{33}$ , discussed in the previous paragraph will remain relevant for a range of bulk anisotropy (although with anisotropy-dependent scaling dimension) and will generally be present in the effective Hamiltonian, unless fine-tuning is done. The stable fixed points are the two trivial ones discussed in the previous paragraph. A non-trivial fixed point appears as an unstable ‘resonance’ critical point. The critical exponents at this non-trivial critical point should vary continuously with bulk anisotropy. However, they are only known at the  $SU(3)$  symmetric point (and at two other points where the non-trivial critical point merges with one of the trivial critical points, using the ‘ $\epsilon$ ’ expansion [2]). Thus we see that the  $SU(3)$ -invariant spin chain has the very special property that the non-trivial critical point is stabilized by a symmetry. This  $SU(3)$ -invariant spin chain (or equivalently  $tJV$  model) would thus provide a convenient model for numerical study of this non-trivial critical point.

It is also interesting to consider a different type of  $SU(3)$  symmetry breaking:  $SU(3) \rightarrow SO(3)$  such that the 3 representation of  $SU(3)$  transforms under the triplet ( $j = 1$ ) representation of  $SO(3)$ . As shown by Itoi and Kato [16], this symmetry-breaking pattern occurs in ordinary  $SU(2)$  spin-1 spin chains with biquadratic as well as bilinear exchange interactions:

$$H = J \sum_j [\cos \theta \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \sin \theta (\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2]. \quad (5.1)$$

The model with  $\theta = \pi/4$  is exactly equivalent to the  $SU(3)$  spin chain. Varying  $\theta$  corresponds to this pattern of  $SU(3)$  symmetry breaking in the continuum limit  $SU(3)_1$  WZW model. Only marginal symmetry-breaking interactions are generated in the effective Hamiltonian. In the case  $\theta > \pi/4$  these can be shown to be marginally irrelevant [16]. The remarkable conclusion is that the  $S = 1$  chain has a gapless phase for all  $\pi/4 < \theta < \pi/2$ . (On the other hand, for  $-\pi/4 < \theta < \pi/4$ , the system goes into the Haldane gap phase.) The effective Hamiltonian of the gapless phase is the  $SU(3)_1$  WZW model, up to logarithmic symmetry-breaking corrections. Now let us consider the effect of this pattern of  $SU(3)$  symmetry breaking in the impurity models. Noting that the 8 representation of  $SU(3)$  decomposes into the direct sum of spin  $j = 2$  and  $j = 1$  representations, with no  $SO(3)$  singlets, we conclude that no relevant or marginal operators are allowed in the effective boundary Hamiltonian at the mixed

critical point, even when the  $SU(3)$  symmetry is broken down to  $SO(3)$ . Thus the boundary critical phenomena that we have elucidated for the  $SU(3)$ -invariant model should also occur in the general bilinear–biquadratic spin-1 chain, with bulk couplings  $\pi/4 < \theta < \pi/2$ . This holds out the possibility of experimental observation of these critical phenomena.

## 6. Connection with the Kondo model and quantum Brownian motion and extension to $SU(n)$

There is clearly a close connection between the RG flows that we have discussed in the  $SU(3)$  spin chain and those in the  $SU(3)$  two-channel Kondo model. As already mentioned below equation (3.17), starting from the case of two equal weak links is equivalent to the RG flow from weak coupling in the Kondo model, with the two decoupled chains on either side of the central spin acting as the two channels. In the continuum limit the correspondence is also clear from the occurrence of the  $SU(3)_2$  WZW model. The mixed critical point in the spin chain corresponds to the non-trivial overscreened fixed point in the Kondo model. (For a discussion of this model, see [14, 15].) In both models this fixed point can be obtained by fusion with the 3 representation operator in the  $SU(3)_2$  WZW model. However, the phase diagram at stronger coupling (beyond the non-trivial critical point) is different in the two models. In the spin chain, at stronger coupling we encounter the (unstable) uniform fixed point and then at infinite coupling the open fixed point. In the Kondo model, on the other hand, the only other fixed point is expected to be the unstable overscreened one at infinite coupling. We also remark that breaking the reflection symmetry, so that one weak link is of different strength than the other, is equivalent to channel symmetry breaking in the two-channel Kondo model. This is a relevant perturbation at the non-trivial fixed point in both cases.

It is also worth remarking in more detail on the connection between the boundary critical phenomena that we have been discussing in the  $SU(3)$  spin chain and that in the model of quantum Brownian motion (QBM) in [4]. The latter model has two massless bosons, defined on the half-line, with boundary sine-Gordon interactions. The  $SU(3)$  spin chain can be regarded as having four left-moving massless bosons on the half-line, corresponding to the central charge 4 in equation (1.4). The  $SU(3)_2$  WZW model can be written as a conformal embedding of two free bosons (corresponding to the maximal Abelian subgroup of  $SU(3)$ ) together with a  $c = 6/5$  conformal field theory which can apparently be regarded as the  $Z_3^{(5)}$  CFT discussed in [4]. This, together with the Potts model, ( $c = 4/5$ ) comprises the two free bosons ( $c = 2$ ) occurring in the QBM model. Since the extra two free bosons of the spin chain do not occur in the effective Hamiltonian if  $SU(3)$  symmetry (or even its  $U(1) \times U(1)$  subgroup) is maintained, there is a correspondence between RG flows in QBM and in the spin chain. The Dirichlet, Neumann,  $Y$  and  $W$  fixed points in the QBM model correspond, respectively, to open, uniform, mixed and new fixed points in the spin chain. In both models all these fixed points can be constructed by fusion with the same operators in the Potts sector starting from the Dirichlet (i.e. open) fixed point.

Finally, we remark that most of the considerations of this paper can be extended to the general case of  $SU(n)$  ‘spin’ chains. After regarding the right-movers as a second branch of left-movers, we can again introduce a conformal embedding:

$$SU(n)_1 \times U(n)_1 = SU(n)_2 \times Z_n \quad (6.1)$$

where  $Z_n$  refers to the  $Z_n$  parafermion conformal field theory. Non-trivial critical points can again be constructed by the fusion method and given a physical interpretation in these lattice models. These fixed points have already been discussed in the context of quantum Brownian motion [3, 4].



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